# Week 14: Beam Buckling

- Euler Buckling
- Effective length for buckling
- Effect of eccentricity

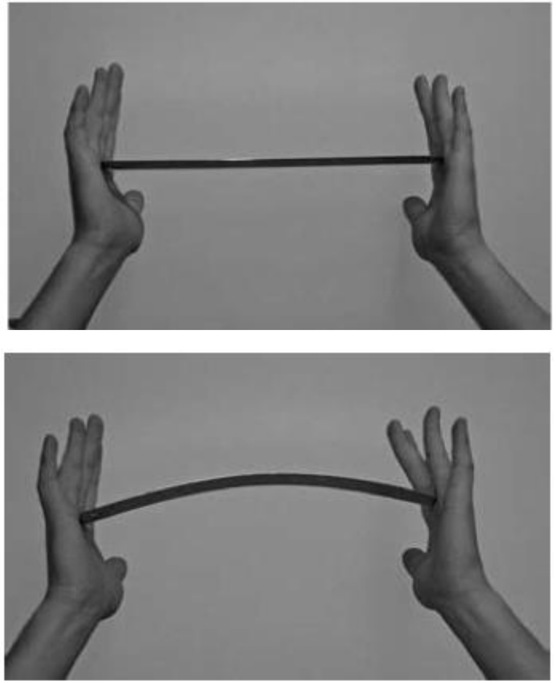
# Stability- different criteria for resisting loads

<u>Strength:</u> the ability of a structure to withstand a load without the development of excessive stress

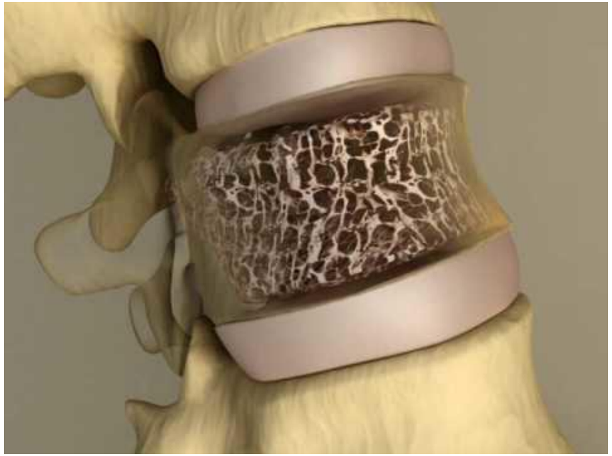
<u>Stiffness:</u> the ability of a structure to withstand a load without developing excessive deformation.

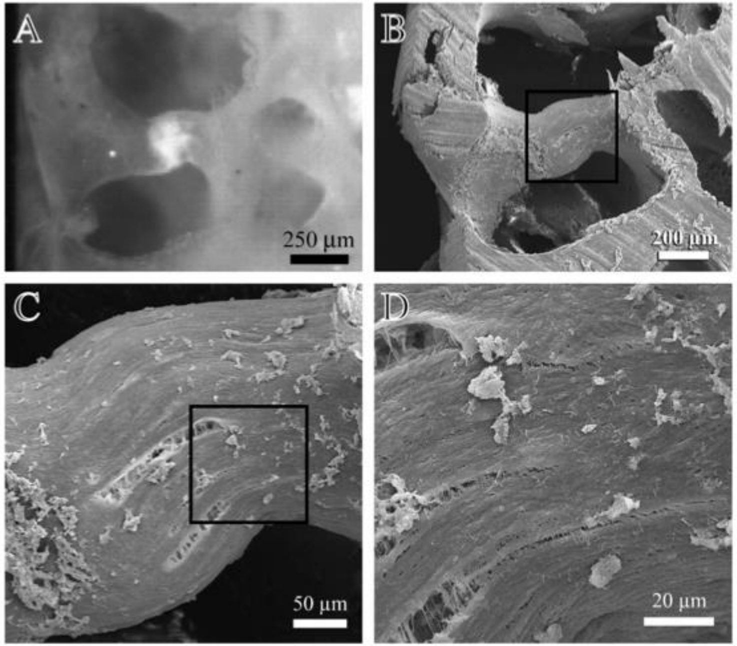
<u>Stability:</u> the ability of a structure to withstand a load without experiencing a sudden change in configuration

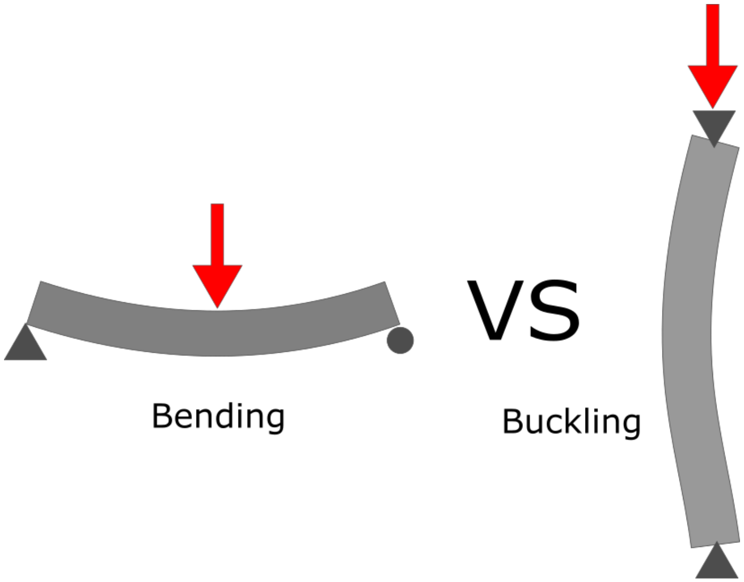
# Buckling

# Buckling is important from the macro to the microscale

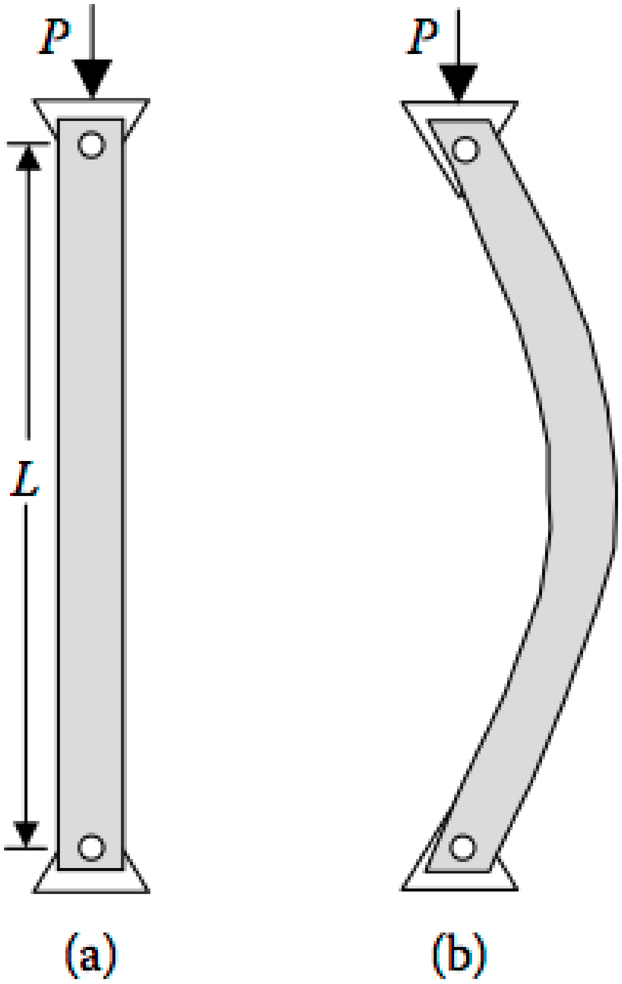







# Buckling

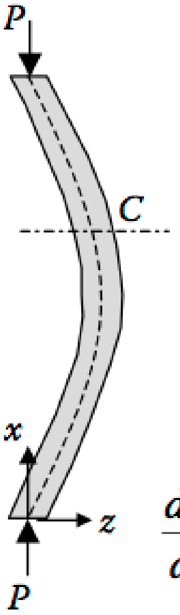
- Buckling is a type of instability that occurs when a beam fails under a compressive load much smaller than the load necessary to reach the yield stress
- In buckling, the failure occurs because the applied load results in a sudden deformation in a perpendicular direction.

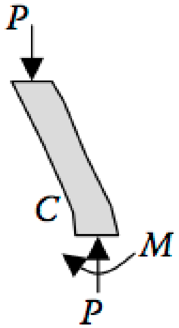



# Euler Buckling

Two regimes of deformation when a beam is loaded in compression:

1. If the axial load on a beam is small, the change in length will be due to compressive strain. $P<P_{cr}$
2. If the axial load on a beam P is larger than the critical load $P_{cr,}$ then the beam becomes instable and a small perturbation will result in buckling of the beam.



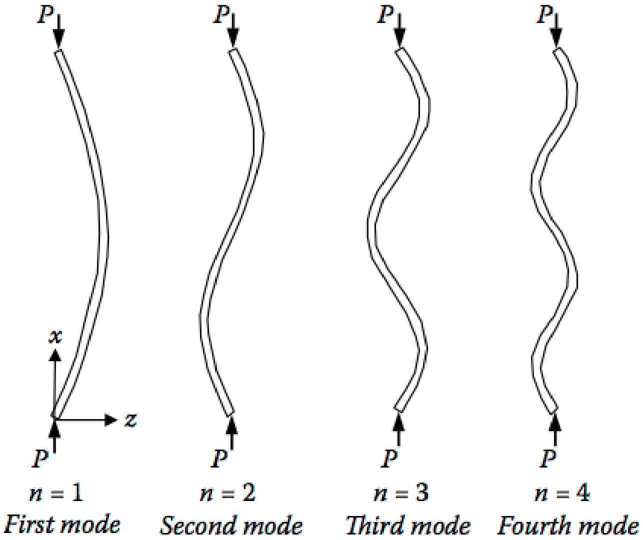




$$\frac{d^2w}{dx^2} = -\frac{M}{EI} = -\frac{P}{EI}w$$

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = 0$$

# Euler Buckling

We can derive the Euler Buckling formula using the method of sections through the buckled beam.




# Euler Buckling

The differential equation has multiple solutions:

This results in multiple bucking modes:

$$P = \frac{n^2 \pi^2 EI}{L^2}$$

# Euler Formula

- For the critical buckling load we get then *Euler's Buckling Formula*:

$$P_{cr} = \frac{\pi^2 EI}{L^2}\,.$$

- With the shape of the buckled beam:

$$w(x) = a\sin(\frac{\pi}{L}x)$$

- The second moment of area (I) should be taken around the axis around which the beam buckles. This is in general the axis with the smallest second moment of area.

# Euler Buckling : Effective length

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

Effective Length $L_e$ as Function of Supports

| End Conditions | Effective Length |
|---|---|
| Fixed-Free | $Le = 2L$ |
| Pinned-Pinned | $Le = L$ |
| Fixed-Pinned | $Le = 0.7L$ |
| Fixed-Fixed | $Le = 0.5L$ |

- The Euler Formula we have derived here only deals with a beam with pinned supports on each end.
- The type of support however greatly influences the critical load and the buckling behavior.
- Euler's formula can be extended towards other types of support by using the concept of the effective length.

# Critical Buckling stress



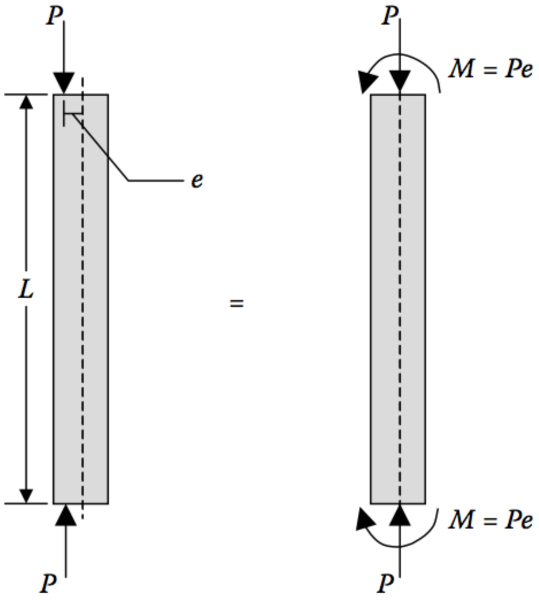
- Critical buckling stress:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{L_e^2 A}$$

- Using the definition of the radius of gyration $r = \sqrt{\frac{I}{A}}$ :

$$\sigma_{cr} = \frac{\pi^2 EAr^2}{L_e^2 A} = \frac{\pi^2 Er^2}{L_e^2} = \frac{\pi^2 E}{(L_e / r)^2}$$

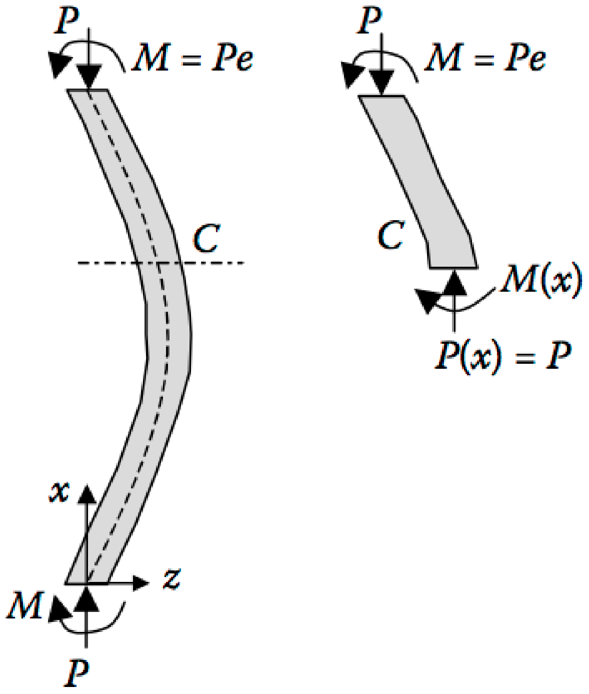
- $L_e/r$ is the slenderness ratio




# Buckling: Effect of eccentricity

- So far, we've looked at beams that were loaded through the centroid of the column
- Often, the load is offset from this axis: eccentric loading
- We calculate the behavior of a beam pinned at both ends with eccentric load.
- We can model the eccentricity with an axial load and a moment at the supports

# Buckling: Effect of eccentricity




- Centric load: P
- Moment: M=P*e
- This means that the beam bends even under small loads without the beam buckling
- We solve the now inhomogeneous differential equation:

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = -\frac{P}{EI} \cdot e$$

# Buckling: Effect of eccentricity

- For the deflection we then get:

$$w(x) = e\left\{\tan\left(\sqrt{\frac{P}{EI}}\cdot\frac{L}{2}\right)\cdot\sin\left(\sqrt{\frac{P}{EI}}\cdot x\right)+\cos\left(\sqrt{\frac{P}{EI}}\cdot x\right)-1\right\}$$

- And for the maximum deflection:

$$w_{max} = w(L/2) = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)-1\right]$$

- The critical bucking load is then

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

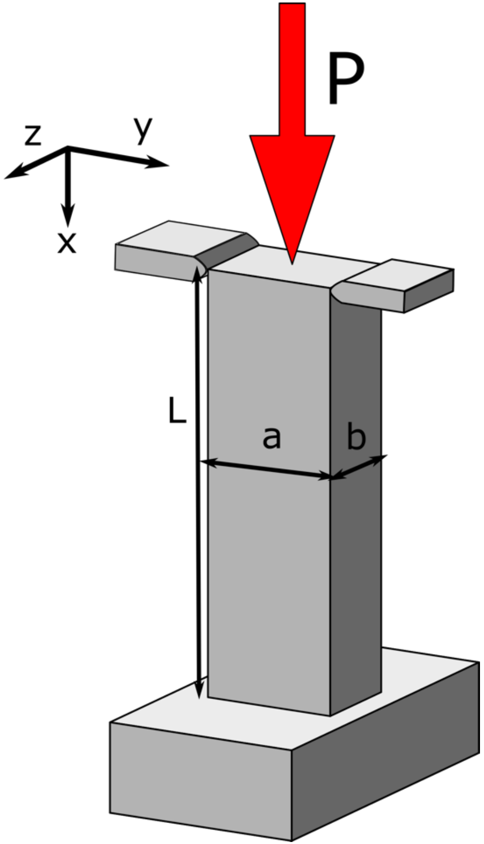
# Buckling: Effect of eccentricity



- Maximum stress in the beam is given by the compressive stress and the bending stress:

$$\sigma_{max} = -\frac{P}{A}\left[1 - \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right]$$

$$\sigma_{max} = -\frac{P}{A}\left[1 - \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{P_{cr}}}\frac{\pi}{2}\right)\right]$$




# Example Buckling

An aluminum column of length *L* and rectangular cross section has a fixed end *B* and supports a centric axial load at *A*.

Two smooth and rounded fixed plates restrain end *A* from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

(a) Determine the ratio *a/b* of the two sides of the cross section corresponding to the most efficient design against buckling.

(b) Design the most efficient cross section for the column, knowing that $L$ = 50 cm, $E$ = 70 GPa, $P$ = 22 kN and that a safety factor of 2.5 is required.